CONTEXTS FOR LEARNING mathematics

Minilessons for Early Addition and Subtraction

A Yearlong Resource

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Contents

Contents *Continued*

Overview

Unlike many of the other units in the *Contexts for Learning Mathematics* series, which consist of two-week sequences of investigations and related minilessons, this unit is meant to be used as a resource of 78 minilessons that you can choose from throughout the year. In contrast to investigations, which constitute the heart of the math workshop, the minilesson is more guided and more explicit, designed to be used at the start of math workshop and to last for ten to fifteen minutes. Each day, no matter what other unit or materials you are using, you might choose a minilesson from this resource to provide your students with experiences to develop efficient computation. You can also use the lessons with small groups of children as you differentiate instruction.

Some of the minilessons in this unit make use of quick images with pictures. We call these "billboards." They are carefully designed pictures that support the development of important strategies for addition and subtraction by building in potentially realizable strategies or constraints (see the example below). Flashed for only a few seconds, they encourage children to give up trying to count each item and instead to use their natural ability to subitize—to perceive small amounts (such as 2, 3, or 4) as units and use them. In this way children are supported to count on, skip-count, use doubles, and make use of the five-structure. For example, the picture of shoes below is structured with the potentially-realizable suggestions of skip-counting by twos or using a double, 5 + 5. It provides a way of looking at 10 as 2 + 2 + 2 + 2 + 2 and as 5 + 5.

Other minilessons in this resource unit make use of the arithmetic rack (*rekenrek* in Dutch)—a calculating frame consisting of two rows of ten beads with two sets of five beads in each row. It was developed by Adri Treffers, a mathematics curriculum researcher at the Freudenthal Institute in the Netherlands. Experience with the arithmetic rack allows young children to build on their innate ability to subitize small groups, and to use a group of five as a unit. When five can be subitized as a whole, it can support understanding 6 as 5 + 1, 8 as 5 + 3, or 4 as 5 – 1. The arithmetic rack also supports the strategies of doubles and near doubles, such as 6 + 7 = 6 + 6 + 1, and making ten, such as 9 + 6 = 10 + 5 (Treffers 1991).

Still other minilessons within this guide use ten-frames and coins (such as dimes and quarters) to develop the recognition and use of landmark numbers to help children use larger groups like tens and twenty-fives. No matter what the tool being used may be—billboards, the arithmetic rack, ten-frames, or coins—each minilesson is crafted as a tightly structured series, or "string," of computation problems designed to encourage children to look to the numbers first, before they decide on a computation strategy. The strings are likely to generate discussion on certain strategies or big ideas underlying an understanding of early number sense, addition and subtraction.

This guide is structured progressively, moving from the use of quick images, to the use of the arithmetic rack, to the use of ten-frames and coins. Although you may not use every minilesson in this resource, you will want to work through it with a developmental progression in mind.

The Mathematical Landscape: Developing Numeracy

To really understand addition and subtraction, children must understand how they are related. Children must have a generalized model of quantity, and understand how a whole is made up of parts, parts which may be "rearranged" (for example, added in a different order). By composing and decomposing parts of a whole, children become able to understand and represent the operations of addition and subtraction.

Once children have constructed these landmarks, the traditional next step has been to have them memorize the basic addition and subtraction facts through repetitive drill and practice, using worksheets and flash cards. Do these instructional strategies work? Is it necessary to memorize facts? If so, how do we help children journey toward this horizon?

The debate in our schools has usually centered on understanding vs. memorization, as if the approaches were dichotomies. Children have been expected either to count on their fingers or to memorize isolated facts. Understanding what it means to add and subtract is necessary before the basic facts can become automatic, but understanding does not necessarily lead to automaticity. In other words, understanding is necessary but not sufficient. Children often develop a good understanding of what it means to add two numbers, and they demonstrate this understanding by showing with their fingers, or with cubes, the numbers they are adding. Even with this understanding, however, they may count three times—each quantity separately, then the total. For example, to figure out 5 + 6, they may initially count from 1 to 5, then 1 to 6, and then combine the sets and start all over again, counting the whole group from 1 to 11. Even when they construct the strategy of counting on from the greater number, they may still rely on counting with their fingers, saying, "Seven, eight, nine, ten, eleven."

While these strategies may be understandable in the beginning of the development of number sense, children cannot be left with only these limited methods for solving addition and subtraction problems. But is the answer memorization of isolated facts? How many facts are there? And how do we help children understand the relationships among facts (e.g., 5 + 6 = 5 + 5 + 1)?

Important Addition Strategies

Children who struggle to commit basic facts to memory often believe that they have to memorize "hundreds" of facts because they have little or no understanding of the relationships among them. Children who commit the facts to memory easily are able to do so because they have constructed relationships among the facts, and between addition and subtraction in general, and they use these relationships as shortcuts. Here are some strategies that are important:

- Double plus or minus—for example,
  6 + 7 = 6 + 6 + 1 (or 7 + 7 – 1) = 13.
- Working with the structure of five—for example,
  6 + 7 = 5 + 1 + 5 + 2 = 10 + 3 = 13.
- Making ten—for example,
  9 + 7 = 10 + 6 = 16.
- Using compensation—for example,
  6 + 8 = 7 + 7 = 14.
- Using known facts—for example,
  6 + 8 = 14, so 7 + 8 must be 14 + 1 = 15.

Memorizing facts with flash cards or through drill and practice on worksheets will not develop recognition of these relationships.

Memorization or Automaticity?

Memorization of basic facts usually refers to committing the results of unrelated operations to memory so that thinking through a computation is unnecessary. Isolated additions and subtractions are practiced one after another as if there were no relationships among them; the emphasis is on recalling the answers. Teaching facts for automaticity, in contrast, relies on thinking. Answers to facts must be automatic, produced in only a few seconds; counting each time to obtain an answer is not acceptable. But thinking about the relationships among the facts is critical. A child who thinks of 9 + 6 as 10 + 5 produces the answer of 15 quickly, but thinking rather than memorization is the focus (although over time these facts are eventually remembered). The issue here is not whether facts should eventually be memorized but how this memorization is achieved: by drill and practice, or by focusing on relationships.

Isn't memorization faster? Interestingly, no! Kamii (1985) compared two first-grade classrooms in the same school. In one, the teacher focused on relationships and worked toward automaticity. In the other, children memorized facts with the help of drill sheets and flash cards. The children in the classroom in which automaticity was the goal significantly outperformed the traditionally taught students in being able to produce correct answers to basic addition facts within three seconds—76 percent compared with 55 percent. Some of the most difficult facts for the traditional students were 8 + 6, 5 + 7, 5 + 8, 9 + 5, and 7 + 6. These were solved easily by the other group with strategies like double plus or minus, working with the structure of fives, and making ten.

When relationships are the focus, there are far fewer facts to remember, and big ideas like compensation, hierarchical inclusion, and part-whole relationships come into play. Also, a child who forgets an answer has a quick way to calculate it.

Hands-on learning has traditionally been viewed as the accepted approach to ensure the development of number sense in young children. In the United States, the manipulatives most commonly used are single objects that can be counted—connecting cubes, bottle caps, counters, or buttons. While these manipulatives have great benefits in the very early stages of counting and modeling problems, they do little to support the development of the important strategies needed for automaticity. In fact, at a certain point they begin to reinforce low-level counting strategies. For example, to solve 6 + 7 with cubes, children need to count out 6, then 7, and then either count on as they combine or (as is most common) count a third time. Because the materials have no built-in structure, they offer little support for the development of alternative strategies.

Building structure into manipulatives is not always beneficial by itself, however. For example, an abacus and base-ten blocks have a base-ten structure built in. The problem with these materials is that while the structures in them are apparent to adults, they are not always apparent to children.

Resnick and Omanson (1987, cited in Gravemeijer 1991) give a beautiful example of children's difficulties when using base-ten materials: A child is looking at six flats (worth 600) and five sticks (worth 50). The teacher asks, "So how much do you think this would be?"

The child answers, touching the hundred blocks (the flats), "100, 200, 300, 400, 500, 600," and continues, touching the ten blocks (the sticks), "700, 800, 900, ten hundred, eleven hundred."

The teacher points to the tens blocks and asks, "Are these worth 100?"

"I count them all together."

"But these [tens] aren't hundreds."

"I am counting these like tens," explains the child.

"OK. But how much would these tens be worth?"

"Oh, 10, 20, 30, 40, 50. . . 50 dollars."

Now the teacher asks about the whole amount: "How much would this [entire display] be worth altogether?"

"600 . . . wait! It's 5 and 6."

"But how much is it altogether? This [hundred] is 6, right?"

"Eleven hundred."

If children have not constructed the big idea of unitizing, they do not see the ten block as one ten; they see it simply as a unit. The use of these materials is based on an empiricist or activity learning theory.

The assumption is that if children just use the materials enough, they will "take in" or "come to see" the arithmetical structure. From a constructivist perspective on learning, we need to ask what it is *the child* is seeing, rather than what it is we expect the child to see.

The materials cannot transmit knowledge; the learner must construct the relationships (Gravemeijer 1991). Streefland (1988) suggests that instead of building adults' mathematical structures into materials, designers look to children's invented alternative strategies as road signs that suggest how to build manipulatives that enable children to realize their own ideas. In this way, structured materials can both support natural development and stretch children's ability to restructure an initial strategy and adopt a better shortcut.

With these goals in mind—supporting and stretching children's natural development—Adri Treffers, a mathematics curriculum researcher at the Freudenthal Institute in Holland, developed a tool called the *rekenrek*. Directly translated, *rekenrek* means calculating frame, or arithmetic rack. He also designed an accompanying didactic to support the natural mathematical development of children—to encourage them to use strategies like double plus or minus, working with the structure of five, compensation, and making tens instead of counting (Treffers 1991).

While the arithmetic rack may seem like an abacus at first glance, it is not based on place value columns and it is not used in that way. Rather, it consists of beads in two rows of ten, each broken into two sets of five. Beads are grouped on the left to represent the amounts being calculated. The five-structure offers visual support (the quantity of five can often be subitized as a whole) and supports the idea that seven comprises five and two; eight, five and three, and so on.

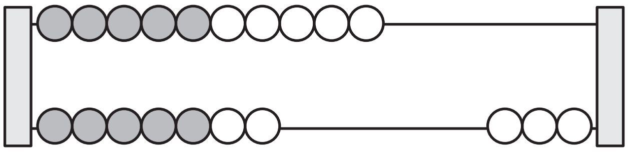

Figure 1

Figure 1 shows an arithmetic rack with 10 + 7 represented. Children might calculate the total as 5 + 5 + 5 + 2, 15 + 2, 7 + 7 + 3, or as 10 + 7. They can also count three times, or count on, if they need to, but the manipulative is likely to encourage children to structure shortcuts. The arithmetic rack also allows for various strategies with subtraction. To remove 9 from 17 on the rack, children could remove 9 beads from the top, leaving 1, plus 7 on the bottom. Or remove the 7 from the bottom and 2 from the top, leaving 8. Or remove the 2 white beads from the bottom and 7 more from the top, leaving 3 on the top and 5 on the bottom. The five- and ten-structures of the apparatus support children in envisioning the removal.

Using Minilessons to Develop Number Sense: An Example

Minilessons are usually done with the whole class together in a meeting area. Young children often sit on a rug; for older students, benches or chairs can be placed in a U-shape. Clustering students together like this, near a chalkboard or whiteboard, is helpful because you will want to provide an opportunity for pair talk at times, and you will need space to represent the strategies that will become the focus of discussion. The problems are done one at a time and learners are asked to determine an answer. Although the emphasis is on the development of mental arithmetic strategies, this does not mean learners have to solve the problems *in* their heads—but it is important for them to do the problem *with* their heads! In other words, encourage children to examine the numbers in the problem and let those numbers guide them in finding clever, efficient ways to reach a solution. The relationships among the problems in each minilesson will support them in doing this. By developing a repertoire of strategies, an understanding of the big ideas underlying why they work, and a variety of ways to model the relations, children are assembling powerful toolboxes for flexible and efficient computation. Enter a class with us and see how this is done.

Each day at the start of math workshop, Linda Jones, a first-grade teacher in Missouri, does a short minilesson on computation strategies. Today, she is working with a class-size arithmetic rack. Covering the rack with a piece of fabric so the children cannot see what she is doing, she moves over to the left 6 beads on the top and 5 beads on the bottom. When she finishes, she removes the fabric briefly for all to get a glimpse, but then covers the rack again in order to discourage counting strategies.

Linda invites the children to share what they saw and how they figured out the total. "What did you see? Turn to your neighbor and tell what you did." After a few moments of these paired discussions, Linda starts a whole-group discussion. "Kenny, what did you see?"

"I saw five red on the top and one white, and five red on the bottom," Kenny replies quickly. But he gets confused trying to explain how many that is, so he goes up to the now uncovered rack and counts the beads from one to eleven.

His classmate Mike agrees with him but explains that he counted on, "Seven, eight, nine, ten, eleven."

"Jessie, how about you?" Linda asks. "You had your hand up quickly. What did you do?"

"I didn't have to count," Jessie reports proudly, "because I knew there were five red on the top and five red on the bottom and that made ten, and one more is eleven."

"Wow, that's a great strategy, isn't it? Maybe some of you might want to try Jessie's strategy on this next one."

It's not important that everyone share a strategy. Linda elicits three different ones, highlights the efficiency of Jessie's, suggests that others may want to try it, and moves on. She covers the rack up and makes 7 and 7 behind the fabric. "OK, here we go." She uncovers the fabric for a quick glimpse. "What do you see? Talk to your neighbor."

"What did you see on the bottom, Andy?"

"I saw five reds and two whites, and I knew that was seven," Andy explains.

Jake interrupts excitedly. "Yup, and on the top, too. Seven and seven."

"And how much is all of that, Jake?" Linda asks.

"Fourteen," Jake says confidently. He continues explaining his thinking. "I took the whites away on the top and the bottom. That gave me five plus five. That was ten. And four more is eleven, twelve . . . thirteen, fourteen." Although he counts the four whites on, he pauses after twelve and says thirteen and fourteen louder, perhaps mentally envisioning the sets of twos. Several children murmur that they did it the same way.

Linda is working toward automaticity here. If children need to, they can count. But she employs several techniques that are likely to stretch children beyond counting. First, she reveals the frame only briefly, then covers it again until they begin to share their strategies. Children who need to count each bead can, and do so (often using their fingers). But the rack supports the use of the five-structure and making ten. Linda highlights this as a terrific shortcut and encourages others to use it. She follows up with the next problem in her string: a good choice of numbers (7 + 7) since 7 can be decomposed into 5 + 2 on the rack and the problem is then easily seen as 10 + 4. Other good choices would be 5 + 8 or 6 + 8, where the five-structure can also be helpful.

During another minilesson later in the year, Linda continues her work. This time the children use their own racks. "So let's get in a circle, and I'll pass the racks out." Linda stands by a large dry-erase board as part of the circle that the children form. She passes out the racks and writes 7 + 6 on the board. Next to that she also draws a representation of an "open rack"—two lines with no beads—while she gives the children time to figure the problem out on their racks. Then she starts the discussion. "Mike, how did you do it? How many reds on top?"

"Five," Mike responds, "and two whites."

Linda draws five red beads on the open rack on the board. "What color shall I use for the whites? White won't show up."

"Blue," several children chorus, and she draws two blue beads next to the five red.

"And it's five reds and one white on the bottom," Mike continues.

Linda completes the drawing and asks, "And so how did you figure out how many it was all together?"

Mike explains that he counted 2, 4, 6 (moving two beads each time), then 7, 8, 9, 10, 11, 12, 13. Linda records his thinking as in Figure 2.

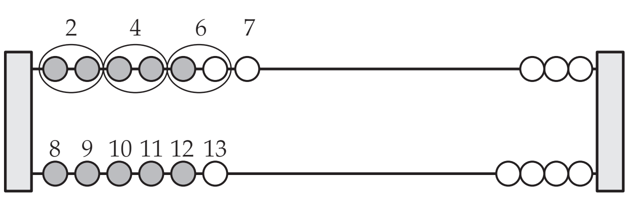

FIGURE 2

Belinda also offers a counting strategy, although she counts on from seven. Linda records her strategy also. *[See Figure 3.]*

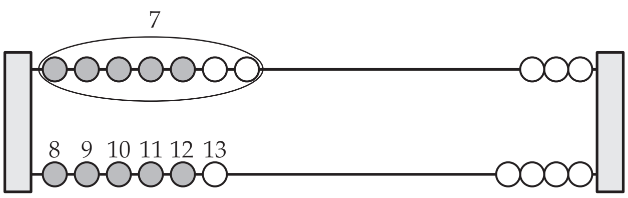

FIGURE 3

"Lorene, what about you?" Linda draws another open rack on which to record Lorene's solution.

"I knew seven plus seven was fourteen, so I just took one away and that made thirteen."

"Wow, Lorene. What a great strategy. Did that make sense to you, Dean?" Linda attempts to pull Dean, who looks puzzled, into the conversation. Dean struggles to paraphrase Lorene's strategy but cannot.

Lorene explains again. "I just knew that seven plus seven was fourteen." Linda draws seven and seven on the open rack. "And then I took one away, and that made thirteen." Linda draws an "X" over one bead to represent Lorene's thinking as in Figure 4.

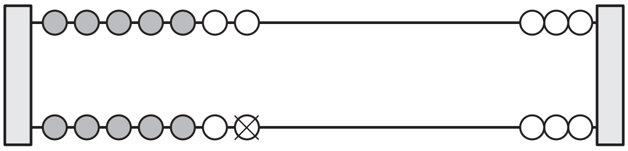

FIGURE 4

"Oh, I get it," Dean smiles; "that's cool." He explains his earlier puzzlement. "I had mine the other way around. I had six on the top and seven on the bottom."

"Does that matter?" Linda puts the question to everyone.

"No," says Kristi. "See, if you take one off the top and put it on the bottom, it's still the same amount." She demonstrates by turning 7 + 6 into 6 + 7. Dean says he agrees with her reasoning.

In this second minilesson, children gain experience with the big idea of the commutative property of addition (6 + 7 = 7 + 6) and, once again, with the logic of compensation—if one addend gains one and the other loses one, the total remains the same. The arithmetic rack supports the use of counting strategies, even counting by twos and counting on, but it also promotes the development of big ideas, and the restructuring of counting strategies in favor of better shortcuts, like using doubles plus or minus.

Although children may initially compute solutions to problems in the minilesson in unsystematic ways, as they share their strategies they have an opportunity to notice and discuss patterns in the string and in the answers. They may become intrigued by the patterns they notice in the problems; they are often impressed with and interested in their classmates' strategies and frequently adopt them when they seem appropriate and efficient.

These young mathematicians are composing and decomposing numbers flexibly as they work. They are inventing their own strategies. They are looking for relationships among the problems. They are looking to the numbers first before they decide on a strategy, and they are automatizing the basic facts.

Linda has developed this ability in her students by focusing on computation each day, with the use of minilessons made up of strings of related problems. She has developed the big ideas and models through investigations, but once this understanding has been constructed, she promotes fluency with computation strategies in minilessons such as the ones in this resource guide.

A Few Words of Caution

As you work with the strings and models in this resource book, it is very important to remember two things. First, honor children's strategies. Accept alternative solutions and explore why they work. Use the models to represent children's strategies and facilitate discussion and reflection on the strategies shared. Sample classroom episodes (titled "Inside One Classroom") are interspersed throughout this unit to help you anticipate what learners might say and do, and to provide you with images of teachers and students at work. The intent is not to get all learners to use the same strategy at the end of the string. That would simply be discovery learning. The strings are crafted to support development, to encourage children to look to the numbers, and to use a variety of strategies helpful for those numbers.

Second, do not use the strings as a recipe that cannot be varied. You will need to be flexible. The strings are designed to encourage discussion and reflection on various strategies important for numeracy. Although the strings have been carefully crafted to support the development of these strategies, they are not foolproof: if the numbers in the string are not sufficient to produce the results intended, you will need to insert additional problems, depending on your students' responses, to provide extra opportunities to explore the ideas and strategies in question. For this reason, each string is accompanied by a Behind the Numbers section describing the string's purpose and how the numbers were chosen. Being aware of the purpose of each string will guide you in determining what type of additional problems are needed. The notes should also be helpful in developing your ability to craft your own strings. Strings are fun both to do and to craft.

References and Resources

Dolk, Maarten and Catherine Twomey Fosnot. 2004a. *Addition and Subtraction Minilessons, Grades PreK–3.* CD-ROM with accompanying facilitator's guide by Antonia Cameron, Sherrin B. Hersch and Catherine Twomey Fosnot. Portsmouth, NH: Heinemann.

———. 2004b. *Fostering Children's Mathematical Development, Grades PreK–3: The Landscape of Learning.* CD-ROM with accompanying facilitator's guide by Sherrin B. Hersch, Antonia Cameron, and Catherine Twomey Fosnot. Portsmouth, NH: Heinemann.

Gravemeijer, Koeno. 1991. An instruction-theoretical reflection on the use of manipulatives. In *Realistic Mathematics Education in Primary School,* ed. Leen Streefland. Utrecht: Utrecht University (CD-ROM series on research in education; 9).

Kamii, Constance. 1985. *Young Children Reinvent Arithmetic.* New York, NY: Teachers College Press.

Resnick, Lauren and S. F. Omanson. 1987. Learning to understand arithmetic. In *Advances in Instructional Psychology* 3, ed. R. Glaser. London: Lawrence Erlbaum.

Streefland, Leen. 1988. *Realistic Instruction of Fractions.* Utrecht: OW and OC, Utrecht University.

Treffers, Adri. 1991. Realistic mathematics education in the Netherlands 1980-1990. In *Realistic Mathematics Education in Primary School,* ed. Leen Streefland. Utrecht: Utrecht University (CD-B series on research in education; 9).