CONTEXTS mathematics FOR LEARNING

Minilessons for Extending Multiplication and Division

A Yearlong Resource

CATHERINE TWOMEY FOSNOT

WILLEM UITTENBOGAARD

*first*hand
An imprint of Heinemann
A division of Reed Elsevier, Inc.
361 Hanover Street
Portsmouth, NH 03801–3912
firsthand.heinemann.com

Harcourt School Publishers
6277 Sea Harbor Drive
Orlando, FL 32887–6777
www.harcourtschool.com

Offices and agents throughout the world

ISBN 0-325-01103-6

ISBN 0-15-360577-4

Library of Congress Cataloging-in-Publication Data
CIP data is on file with the Library of Congress

Printed in the United States of America on acid-free paper

11 10 09 08 07 ML 1 2 3 4 5 6

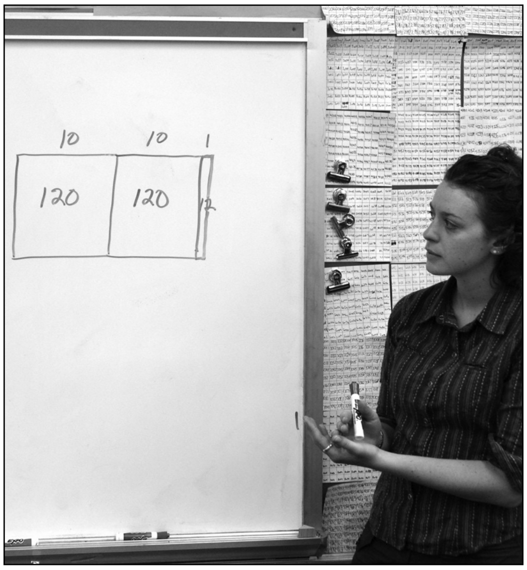
Acknowledgements

Photography

Herbert Seignoret
Mathematics in the City, City College of New York

Contents

Contents *Continued*



Overview

Unlike many of the other units in the *Contexts for Learning Mathematics* series, which consist of two-week sequences of investigations and related minilessons, this unit is meant to be used as a resource of 77 minilessons that you can choose from throughout the year. In contrast to investigations, which constitute the heart of the math workshop, the minilesson is more guided and more explicit, designed to be used at the start of math workshop and to last for ten to fifteen minutes.

This guide is structured progressively, moving from the use of graph paper arrays, to the introduction of the open array but with graph paper provided as support, to the eventual use of the open array solely to represent students' strategies for multiplication and division. The latter section of the guide makes use of the number line and the ratio table instead of the array to support proportional reasoning but includes context to help students realize what they are doing. Although you may not use every minilesson in this resource, you will want to work through it with a developmental progression in mind.

The minilessons in this unit are crafted as groups of computation problems designed to encourage students to look to the numbers first, before they decide on a computation strategy. The minilessons will support your students in developing a variety of mental math strategies as well as an understanding of the traditional algorithms. These tightly structured series, or "strings," of problems are likely to generate discussion of certain strategies and big ideas underlying an understanding of multiplication and division. They can also be used with small groups of students as you differentiate instruction.

The Mathematical Landscape: Developing Numeracy

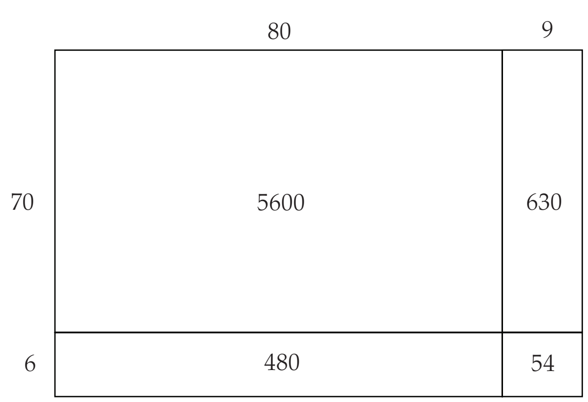
Before you begin working with this unit, try an experiment. Calculate 76×89. Don't read on until you have an answer.

If you are like most people who are a product of the American school system, you probably got a pencil and paper, wrote the numbers down in columns, multiplied 6×89 (starting with 6×9, regrouping the 5 tens and adding it to the product of 6×8, for a partial product of 534), then multiplied by the 7 (starting with 7×9, regrouping the 6 tens, adding it to the product of 7×8, then recording a zero to reflect place value, for a partial product of 6230), and finally added the partial products (starting with the units), for a total of 6764. To check yourself, you probably went back and repeated the same actions and calculations; if you got the same answer twice, you assumed your calculations were correct.

Now take out a piece of graph paper and draw a rectangle that represents a 76 × 89 array. See if you can find the smaller rectangular arrays inside this big one that represent the steps you did as you calculated. If this is difficult for you, the way the algorithm was taught to you has worked against your own conceptual understanding of multiplication.

Students make any number of place value errors in calculating the answer to each of these separate steps, either in regrouping the tens or in lining up the numbers. Why? Just think how nonsensical these steps must seem. Although multiplication is usually introduced in grade three, fourth and fifth graders are often still struggling to understand that 76 × 89 actually means 76 groups of 89—or when envisioned in an array, 76 rows of units (such as square tiles) with 89 units (tiles) in each row. Because students treat the numbers in each step of the problem as digits, they lose sight of the quantities they are actually multiplying.

Liping Ma (1999) compared the way Chinese and American teachers think about and teach the multiplication algorithm and how they work with students who make place value mistakes. Most Chinese teachers approach the teaching of the multiplication algorithm conceptually. They explain the distributive property and break the problem up into the component problems: $76 \times 89 = (70 + 6) \times (80 + 9) = (6 \times 9) + (6 \times 80) + (70 \times 9) + (70 \times 80) = 54 + 480 + 630 + 5{,}600$. Once this conceptual understanding is developed, they compare the steps in the algorithm with the component parts in the equation. (The diagram shows these rectangles, representing partial products, within the larger array of 76 × 89.)



In contrast, 70 percent of American teachers teach the algorithm as a series of procedures and interpret students' errors as a problem with regrouping and lining up. They remind students of the "rules," that they are multiplying by tens and therefore need to move their answer to the next column. To help students follow the "rules" correctly, they often use lined paper and suggest that students use zero as a placeholder (Ma 1999). Sometimes they even resort to teaching the lattice method (an Italian method introduced to Europe by Fibonacci in 1202, and now included in some U.S. curricula) to ensure that fewer place value mistakes will be made.

When students are invited to invent their own ways to decompose the problem—when they have not been taught the algorithm—they usually employ a form of the distributive property naturally! Constance Kamii (1993) found that when she asked students to solve double-digit multiplication problems like 76 × 89, most students began with the largest component. For example, they would break 76 × 89 into 70 × 80 and 6 × 80 and then add the remaining 70 × 9 and 6 × 9. The algorithm, though, requires starting with the units. Kamii, in fact, suggests that teaching algorithms can be harmful to students as they have to give up their own meaning-making just as they are constructing the distributive property in order to adopt the teacher's procedures.

Let's listen in as Sophie, a third grader, tries to make sense of the algorithm in a discussion with Willem Uittenbogaard. They are talking about candies that are sold in rolls of 10.

"How many candies would you have if you had 3 rolls, or 5 rolls, or 7 rolls, or 10 rolls, or 11 rolls?" Willem asks. Sophie easily multiplies by 10 each time. Together they make a table to show the results.

"What if each roll had 11 candies?" Willem then asks. Once again, Sophie responds quickly: "In 3 rolls there would be 33; in 5 rolls, 55; in 7, 77; in 10, 110." But then she pauses. "In 11 . . . I'm not sure. I need paper and pencil." She writes 11 × 11 in column fashion and attempts to perform the algorithm, but she makes a place value error:

$$\begin{array}{r} 11 \\ \times\ 11 \\ \hline 22 \end{array}$$

Looking at her answer of 22, she says, "That can't be." Willem says, "You already know that 10 rolls have 110." Sophie nods, but does not use this information. She writes "11" eleven times in column fashion, and then adds this long column to produce 121. Willem confirms her answer and asks, "What about in 12 rolls?" Once again Sophie asks for paper and pencil and writes:

$$\begin{array}{r} 11 \\ \underline{\times 12} \\ 22 \\ \underline{11} \\ 33 \end{array}$$

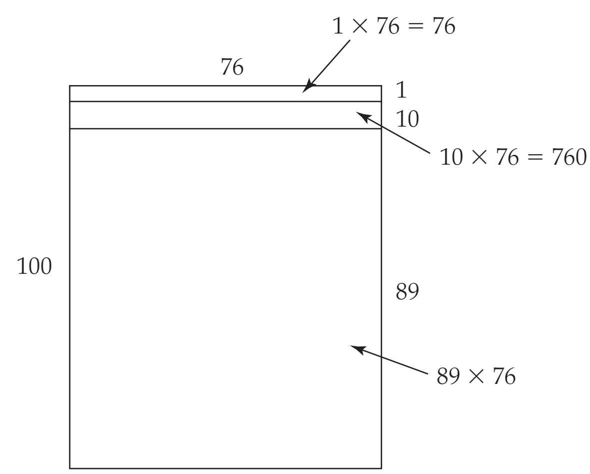
"Can't be." Again puzzled, Sophie returns to her long column addition and adds another 11 on top. This time, however, she does not add each column but crosses out the 121 and writes down 132, obviously adding 11 to 121 in her head.

Sophie understands that this situation calls for multiplication. She understands that it can be solved by repeated addition. She knows that 10 rolls have 110 candies. She can multiply by 10 easily. She demonstrates that she is aware of the distributive property when she adds one more 11 to 121 (which she even does mentally). She also has a good enough sense of number to know that her answers of 22 and 33 can't be right. Her problem is that the algorithm is getting in the way of her sense-making. She treats the digit 1 in 12 as a unit rather than a 10. She is performing a series of memorized steps (incorrectly), rather than trusting in her own mathematical sense.

It is probably true that if American schools taught the algorithm conceptually rather than procedurally, students would develop a clearer understanding of the process. Teachers could, in fact, build a bridge from students' invented solutions to the algorithm by using the distributive property and rectangular arrays. With Sophie, Willem could write out $12 \times 11 = (2 \times 1) + (2 \times 10) + (10 \times 1) + (10 \times 10)$, build an array, and then look at the pieces of the algorithm, picking them out within the larger 12×11 array. But in today's world, do we want Sophie to have to rely on paper and pencil, or do we want to help her trust in her own mathematical sense and add another 11 to 121? She can do this in her head. Is the algorithm the fastest, most efficient way to compute? When are algorithms helpful? What does it mean to compute with number sense? How would a mathematician solve 76×89?

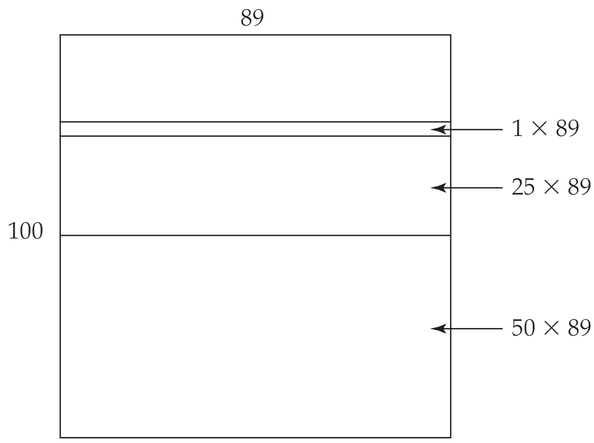
Ann Dowker (1992) asked 44 mathematicians to estimate several typical multiplication and division computation problems, one of which was 76×89, and assessed their strategies. Only 4 percent of the responses represented the use of the standard algorithms. The mathematicians looked at the numbers first and then found elegant, efficient strategies that seemed appropriate for the numbers. They made the numbers friendly (often by using landmark numbers), and they played with relationships. Interestingly, they also varied their strategies, sometimes using different strategies to solve the same problems on different days! Most important, they found the process creative and enjoyable.

A common strategy for 76×89 was $(100 \times 76) - (11 \times 76) = 7600 - 760 - 76 = 6764$ (see the rectangular array below). The commutative and distributive properties were employed to make the problem friendlier.



$7600 - 760 - 76 = 89 \times 76 = 6{,}764$

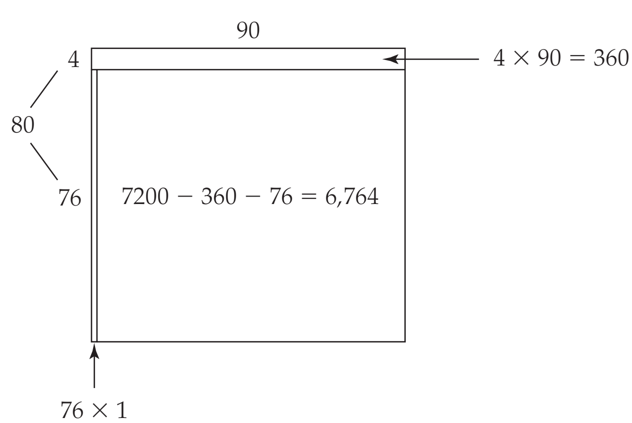
There are many other ways to do this problem. Halving helps. If $100 \times 89 = 8900$, then half of that equals 4450, which is therefore equal to 50×89. To find 25×89, we take half of 4450, which equals 2225. Adding these together, we get $4450 + 2225 = 6675 = 75 \times 89$. Now we just need to add one more 89 to get an answer of 6764. (See the array that follows.)



$75 \times 89 = 6{,}675$
$75 \times 89 = 6{,}675 + 89 = 6{,}764$

Using money sense or fractions is even nicer. Obviously 76 is close to 75, which when thought of in relation to a dollar is ¾, or 75 cents. So ¾ of $89 = 3 \times 22.25 = 66.75$. Since we divided 75 by 100 to get the original ¾, we now have to multiply 66.75 by 100, which equals 6675, the answer to 75×89. Now we need one more 89 to make 76×89: $6675 + 89 = 6764$.

Or we could make the problem friendly by first working with landmark numbers. We could take $80 \times 90 = 7200$, then subtract 4×90. We now have $76 \times 90 = 6840$. To get to 76×89 we need to subtract one more group of 76. Thus, the final answer is 6764. (See the array below.)



Note how all these alternative, creative ways have far fewer steps than the algorithm and can be done more quickly. Some can be done mentally; others may require paper and pencil to keep track. Playing with numbers like this is based on a deep understanding of number, landmark numbers, and operations, and it characterizes true number sense.

Algorithms can nevertheless be helpful, particularly when multiplying or dividing large, unfriendly numbers of four, five, or six digits. And the beauty of the algorithms is that they are generalizable procedures—they work for any numbers, even large, unfriendly ones. But in today's world, isn't that when we take out the calculator anyway? If we have to reach for paper and pencil to perform the arithmetic, why not reach for the calculator? In most real-world situations, the handheld calculator has replaced paper-and-pencil algorithms.

Does the advance of the calculator mean that students don't need to know how to calculate? Of course not. To be successful in today's world, students need a deep conceptual understanding of mathematics. They will be bombarded with numbers, statistics, advertisements, and data every day—on the radio, on television, and in newspapers. They will need good mental ability and good number sense in order to evaluate advertising claims, estimate quantities, efficiently calculate numbers and judge whether these calculations are reasonable, add up restaurant checks and determine equal shares, interpret data and statistics, and so on. Students need a much deeper sense of number and operation than ever before—one that includes algorithms, but emphasizes mental arithmetic and a repertoire of strategies that allows them to both estimate and make exact calculations mentally.

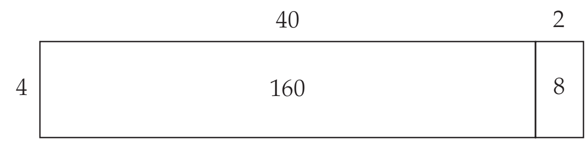
Depending on the numbers, the algorithm is often slower than other mental arithmetic strategies. It only seems faster to most adults because they have used it for many years. The procedures have become habits that require little thinking. Calculating with number sense as a mathematician means having many strategies available and looking to the numbers first *before* choosing a strategy. How do we, as teachers, develop students' ability to do this? How do we engage them in learning to be young mathematicians at work?

Using Minilessons to Develop Number Sense: An Example

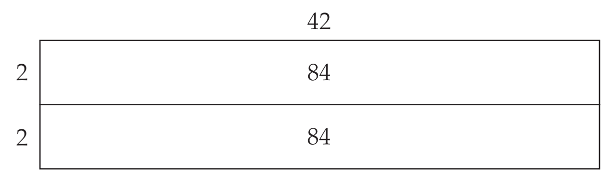
Minilessons are usually done with the whole class together in the meeting area. Young students often sit on a rug; older students can sit on benches placed in a U-shape. Clustering students together like this, near a chalkboard or white board, is helpful because you will want to provide an opportunity for pair talk at times, and you will need space to represent the strategies that will become the focus of discussion. The problems are written one at a time and students are asked to determine an answer. Although the emphasis is on the development of mental arithmetic strategies, this does not mean students have to solve the problems *in* their heads—but it is important for them to do the problem *with* their heads! In other words, encourage students to examine the numbers in the problem and let those numbers guide them in finding clever, efficient ways to reach a solution. The relationships among the problems in each minilesson will support students in doing this. By developing a repertoire of strategies, an understanding of the big ideas underlying why they work, and a variety of ways to model the relations, students are creating powerful toolboxes for flexible and efficient computation. Enter a classroom with us and see how this is done.

Each day at the start of math workshop, Miki Jensen, a fourth-grade teacher in New York City, does a short minilesson on computation strategies. She usually chooses a string of six to eight related problems (like the ones provided in this resource unit) and asks the students to solve them, one at a time, and share their strategies with each other. She allows her students to construct their own strategies for working with numbers in ways that make sense to them. Posted on a strategy wall nearby are signs the students have made throughout the year as they developed a repertoire of strategies for multiplication and division. The signs read, "Make use of money," "Halve and double," "Make use of ten-times," and "When dividing, simplify first." On the chalkboard today as we enter the classroom are Miki's first four problems: 2 × 3, 2 × 30, 4 × 4, and 4 × 40. The students are now engaged in a discussion of the fifth problem, 4 × 42.

Diana, a student in Miki's class, is explaining how she solved the problem: "I know that 4 times 40 is 160, and 4 times 2 is 8. I added and got 168." Miki represents her strategy on an open array:



Diana's math partner Linda, who is sitting next to her, agrees with this answer but used a different strategy: "I just doubled 84. That was 2 times so 4 times is just double that." Miki draws this strategy on an open array as well:

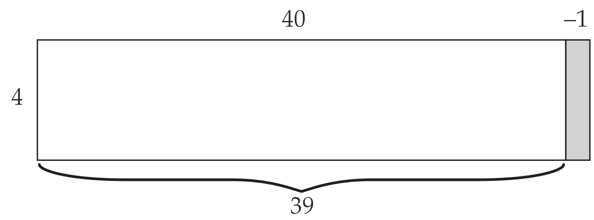


"How many people did this like Linda?" Miki asks, and a few hands go up. "How many of you solved it like Diana?" More hands go up this time. "OK. Turn to the person next to you and convince them that both strategies work." After several minutes of pair talk, Miki asks if there are questions for Diana or Linda. When she sees that there are none, she goes on with her string, writing the next problem, 4 × 39.

"This one isn't so friendly." Miki teases the class with a smile but also offers encouragement: "Take your time. You can use pencil and paper to keep track, but look for a nice efficient way to solve this problem. Show me with a "thumbs-up" signal when you have had enough time." Miki waits until she sees most thumbs up and then she begins discussion. "Alana?"

"I did 4 times 30. That was 120," Alana explains. "Then I did 4 times 9. That was 36. So I added 120 and 36 and got…156." Miki draws an array to represent the partial products.

C.J. nods his head in agreement but says, "I have a shorter way. I used the earlier problem, 4 × 40, and just subtracted 4!" Miki draws the open array shown on page 10 to represent C.J.'s steps and help the other students visualize the parts. "That's pretty neat, isn't it? Who can explain C.J.'s strategy? Diana?"



These young mathematicians are composing and decomposing numbers flexibly as they multiply. They are inventing their own strategies. They are looking for relationships among the problems. They are looking at the numbers first before they decide on a strategy. Students don't do this automatically. Miki has promoted the development of this ability in her students by focusing on computation during minilessons with strings of related problems every day. She has developed the big ideas and models through investigations, but once this understanding has been constructed, she builds fluency with computation strategies in minilessons such as this one.

Using Models during Minilessons

As you work with the minilessons from this resource book, you will want to use models to depict students' strategies. Arrays, ratio tables, and number lines are most helpful for multiplication and division. Representing computation strategies with mathematical models provides students with images for discussion and supports the development of the various strategies for computational fluency, but only if the models are understood. Once the model has been introduced as a representation of a realistic situation, you can use it to record the computation strategies that students share with the class. Doing so will enable students, over time, to use the models as powerful tools to think with.

Note: This unit assumes that the models have already been developed with realistic situations and rich investigations. In this series, the *Muffles' Truffles* unit can be used to develop the array model and *The Big Dinner* unit to develop the ratio table. If your students do not fully understand these models, you may find it beneficial to use these units first. Similarly, since the open number line is the primary model used for addition and subtraction in this series, it is developed in units for earlier grades, *Measuring for the Art Show* and *Ages and Timelines.*

A Few Words of Caution

As you work with the strings and models in this resource book, it is very important to remember two things. First, honor students' strategies. Accept alternative solutions and explore why they work. Use the models to represent students' strategies and facilitate discussion and reflection on the strategies shared. Sample classroom episodes (titled "Inside One Classroom") are interspersed throughout the guide to help you anticipate what learners might say and do and to provide you with images of teachers and students at work. The intent is not to get all learners to use the same strategy at the end of the string. That would simply be discovery learning. The strings are crafted to support development of computational fluency, to encourage students to look to the numbers and to use a variety of strategies helpful for working with those numbers.

Secondly, do not use the string as a recipe that cannot be varied. You will need to be flexible. The strings are designed to encourage discussion and reflection on various strategies important for numeracy. Although the strings have been carefully crafted to support the development of these strategies, they are not foolproof: if the numbers in the string are not sufficient to produce the results intended, you will need to insert additional problems, depending on your students' responses, to support them further in developing the intended ideas. For this reason, most of the strings are accompanied by a Behind the Numbers section describing the string's purpose and how the numbers were chosen. Being aware of the purpose of each string will guide you in determining what types of additional problems to add. These sections should also be helpful in developing your ability to craft your own strings. Strings are fun both to craft and to solve.

References and Resources

Dolk, Maarten, and Catherine Twomey Fosnot. 2005a. *Fostering Children's Mathematical Development, Grades 3–5: The Landscape of Learning.* CD-ROM with accompanying facilitator's guide by Sherrin B. Hersch, Catherine Twomey Fosnot, and Antonia Cameron. Portsmouth, NH: Heinemann.

———. 2005b. *Multiplication and Division Minilessons, Grades 3–5.* CD-ROM with accompanying facilitator's guide by Antonia Cameron, Carol Mosesson Teig, Sherrin B. Hersch, and Catherine Twomey Fosnot. Portsmouth, NH: Heinemann.

Dowker, Ann. 1992. Computational Estimation Strategies of Professional Mathematicians. *Journal for Research in Mathematics Education* 23(1): 45–55.

Kamii, Constance (with Sally Jones Livingston). 1993. *Young Children Continue to Reinvent Arithmetic: Third Grade.* New York, NY: Teachers College Press.

Ma, Liping. 1999. *Knowing and Teaching Elementary Mathematics.* Mahwah, NJ. Lawrence Erlbaum Associates, Inc.